Coping with NP-completeness: Exact Algorithms

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Exact algorithms or intelligent exhaustive search: finding an optimal solution without going through all candidate solutions

Outline

1. 3-Satisfiability
   - Backtracking
   - Local Search
2. Traveling Salesman Problem
   - Dynamic Programming
   - Branch-and-bound

3-Satisfiability (3-SAT)

Input: A set of clauses, each containing at most three literals (that is, a 3-CNF formula).

Output: Find a satisfying assignment (if exists).

Examples

- The formula

$$(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})$$

  is satisfiable: set $x = y = z = 1$ or $x = 1, y = z = 0$.

- The formula

$$(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{x})(\overline{x} \vee \overline{y} \vee \overline{z})$$

  is unsatisfiable.

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A brute force search algorithm checking satisfiability of a 3-CNF formula F with $\boldsymbol{n}$ variables, goes through all assignments and has running time $O(|F| \cdot 2^n)$.

Goal

Avoid going through all 2^n assignments

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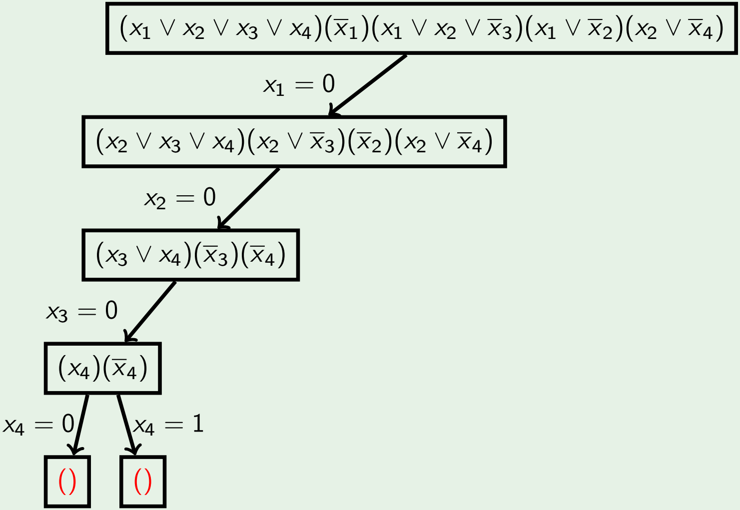
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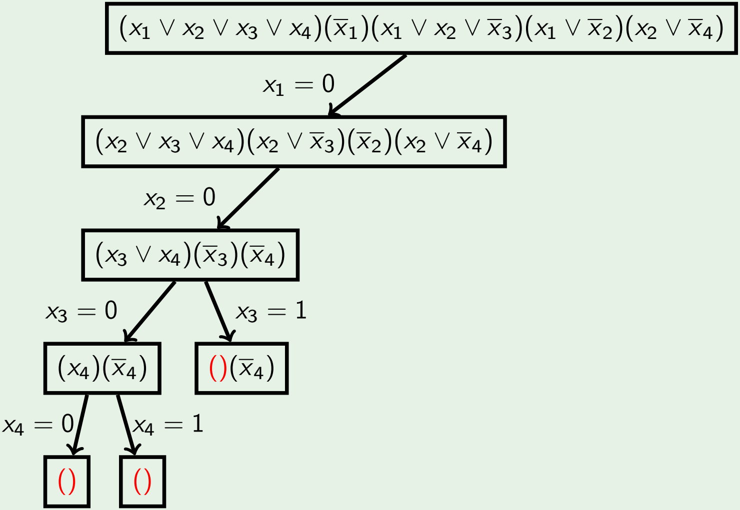
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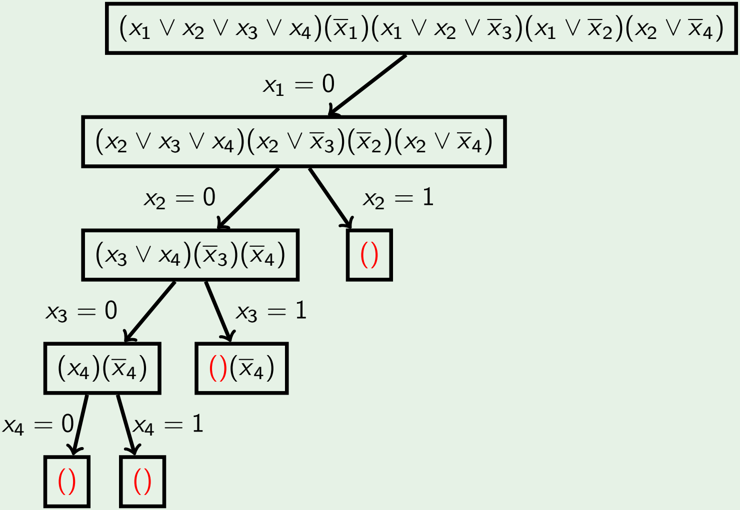
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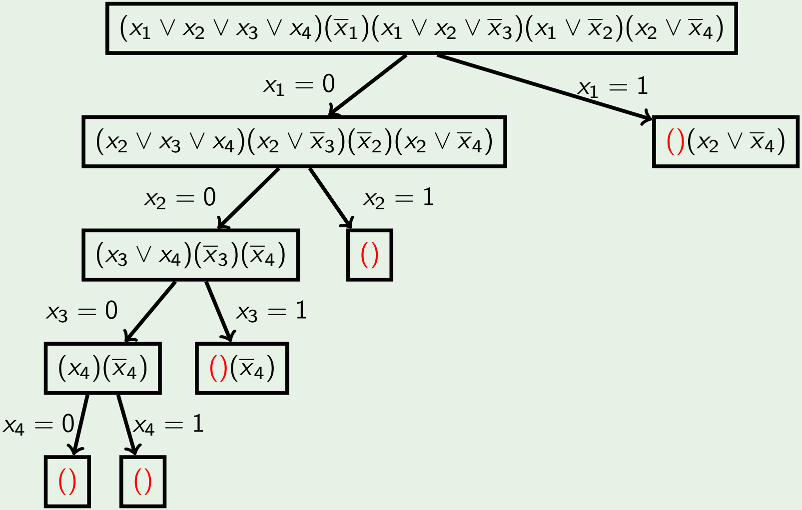
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- When we realize that a branch is dead (cannot be extended to a solution), we immediately cut it

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- SAT-solvers use tricky heuristics to choose a variable to branch on and to simplify a formula before branching
- Another commonly used technique is local search — will consider it in the next part

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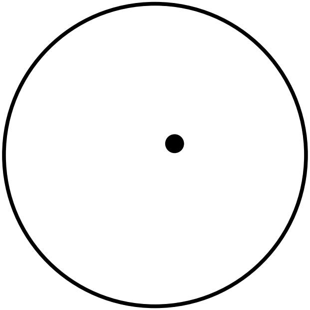
Main Idea of Local Search

- Start with a candidate solution



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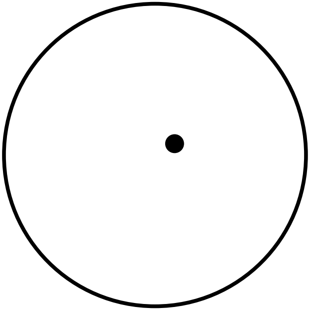

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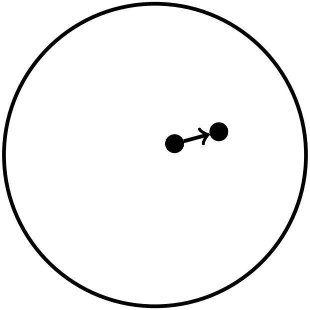

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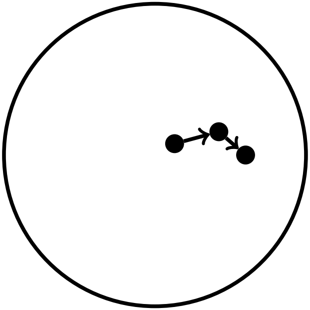

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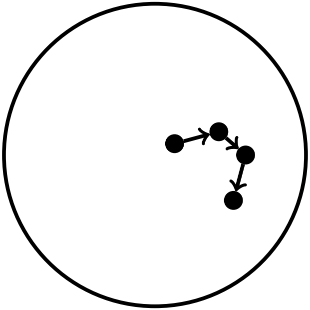

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- A candidate solution is a truth assignment to these variables, that is, a vector from $\{0, 1\}^n$

Definition

Hamming distance (or just distance) between two assignments $\alpha, \beta \in \{0,1\}^n$ is the number of bits where they differ:
$\mathrm{dist}(\alpha, \beta) = |\{i \colon \alpha_i \neq \beta_i\}|$.

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Hamming ball with center $\alpha \in \{0,1\}^n$ and radius r, denoted by $\mathcal{H}(\alpha, r)$, is the set of all truth assignments from $\{0,1\}^n$ at distance at most r from α.

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- $\mathcal{H}(1011, 2) = \{1011, 0011, 1111, 1001, 1010, 0111, 0001, 0010, 1101, 1110, 1000\}$

Searching a Ball for a Solution

Lemma

Assume that $\mathcal{H}(\alpha, r)$ contains a satisfying assignment β for F. We can then find a (possibly different) satisfying assignment in time $O(|F| \cdot 3^r)$.

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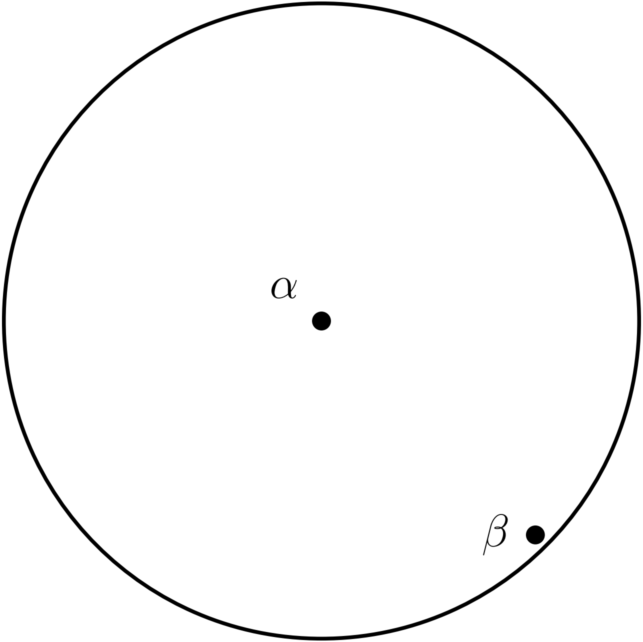
- Hence there are at most 3^r recursive calls $\square$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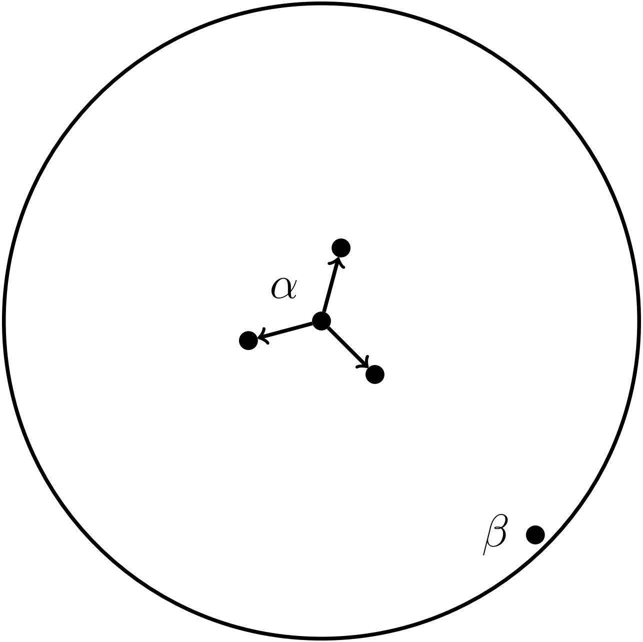
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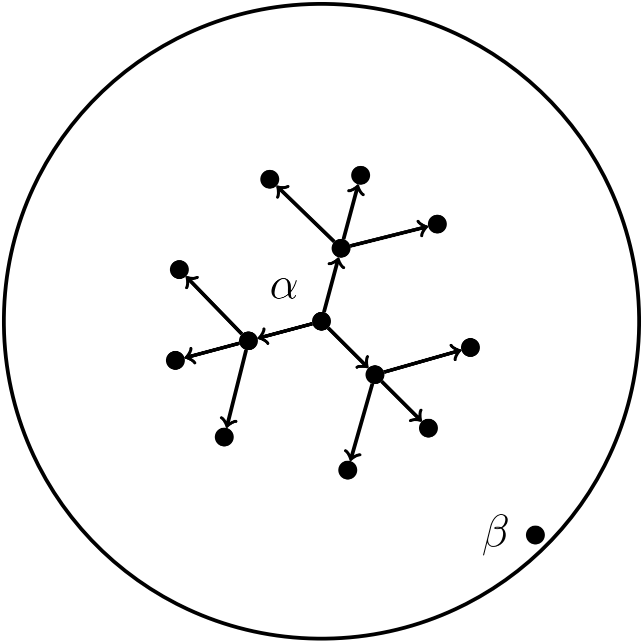

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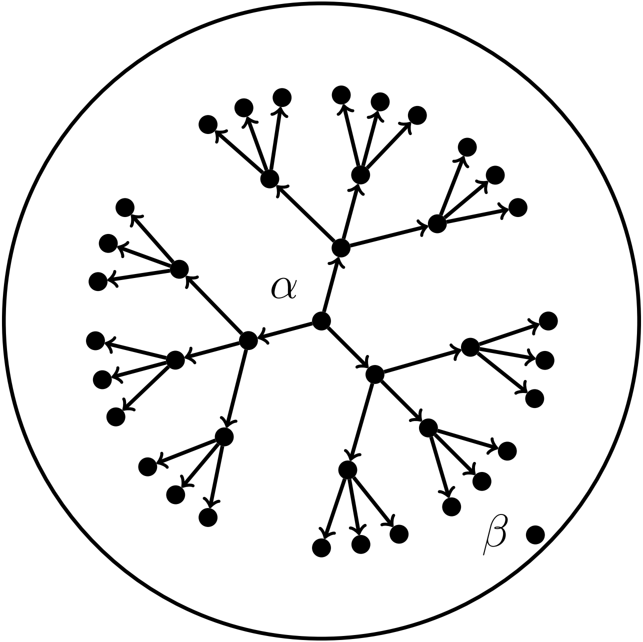

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- Thus, it suffices to make two calls: $\texttt{CheckBall}(F, 11\dots1, n/2)$ and $\texttt{CheckBall}(F, 00\dots0, n/2)$

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- On the other hand, it is exponentially faster than a brute force search algorithm that goes through all 2^n truth assignments!

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Traveling salesman problem (TSP)

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Output: A cycle that visits each vertex exactly once and has total weight at most b.

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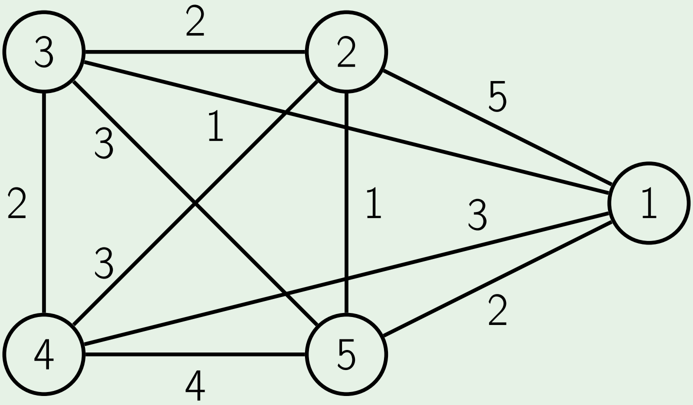
It will be convenient to assume that vertices are integers from 1 to n and that the salesman starts his trip in (and also returns back to) vertex 1.

Example



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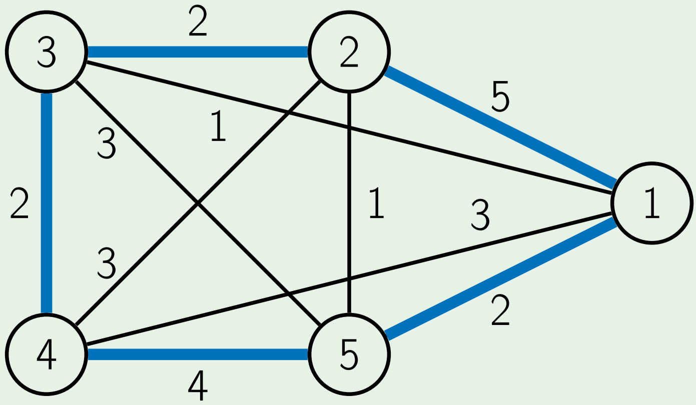
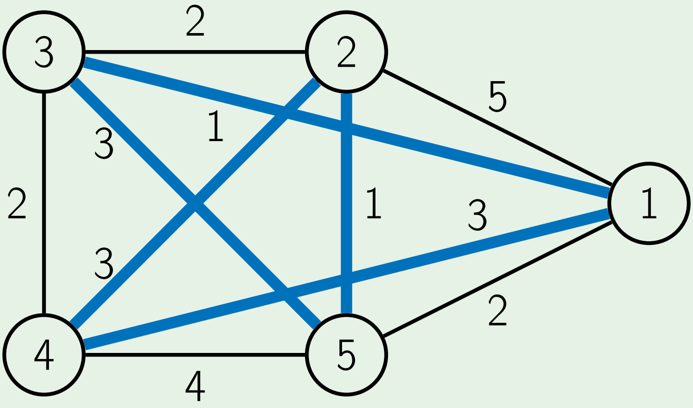


length: 15

Example



length: 11

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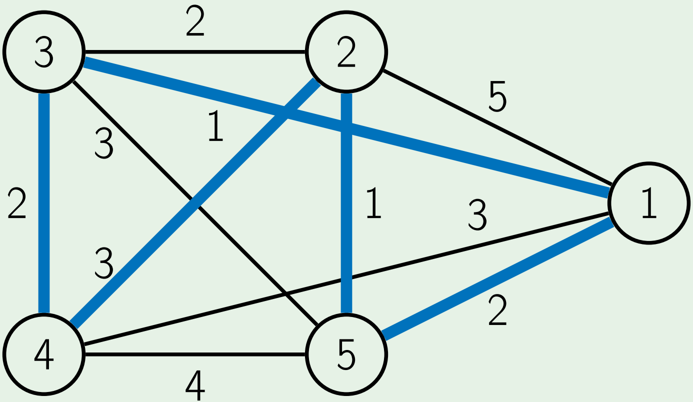

length: 9

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This part

- Use dynamic programming to solve TSP in $O(n^2 \cdot 2^n)$
- The running time is exponential, but is much better than $(n-1)!$.

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- A subproblem refers to a partial solution
- A reasonable partial solution in case of TSP is the initial part of a cycle
- To continue building a cycle, we need to know the last vertex as well as the set of already visited vertices

Subproblems

- For a subset of vertices $S \subseteq \{1, \ldots, n\}$ containing the vertex 1 and a vertex $i \in S$, let $C(S, i)$ be the length of the shortest path that starts at 1, ends at i and visits all vertices from S exactly once

Subproblems

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- $C(\{1\}, 1) = 0$ and $C(S, 1) = +\infty$ when $|S| > 1$

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- Consider the second-to-last vertex j on the required shortest path from 1 to i visiting all vertices from S
- The subpath from 1 to j is the shortest one visiting all vertices from $S - \{i\}$ exactly once
- Hence $C(S, i) = \min\{C(S - \{i\}, j) + d_{ji}\}$, where the minimum is over all $j \in S$ such that $j \neq i$

Order of Subproblems

- Need to process all subsets $S \subseteq \{1, \ldots, n\}$ in an order that guarantees that when computing the value of $C(S, i)$, the values of $C(S - \{i\}, j)$ have already been computed

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- For example, we can process subsets in order of increasing size

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for s from 2 to n:
  for all 1 ∈ S ⊆ {1,...,n} of size s:
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    for all $i \in S$, $i \neq 1$:
      for all $j \in S$, $j \neq i$:
        $C(S,i) \leftarrow \min\{C(S,i), C(S-\{i\},j)+d_{ji}\}$

TSP(G)

```
C({1}, 1) ← 0
for s from 2 to n:
  for all 1 ∈ S ⊆ {1, ..., n} of size s:
    C(S, 1) ← +∞
    for all i ∈ S, i ≠ 1:
      for all j ∈ S, j ≠ i:
        C(S, i) ← min{C(S, i), C(S − {i}, j) + d_ji}
return min_i{C({1, ..., n}, i) + d_i,1}
```

$C(\{1\},1) \leftarrow 0$
for s from 2 to n:
  for all $1 \in S \subseteq \{1,\ldots,n\}$ of size s:
    $C(S,1) \leftarrow +\infty$
    for all $i \in S$, $i \neq 1$:
      for all $j \in S$, $j \neq i$:
        $C(S,i) \leftarrow \min\{C(S,i), C(S-\{i\},j)+d_{ji}\}$
return $\min_i\{C(\{1,\ldots,n\},i) + d_{i,1}\}$

Implementation Remark

- How to iterate through all subsets of $\{1, \ldots, n\}$?

Implementation Remark

- How to iterate through all subsets of $\{1, \ldots, n\}$?
- There is a natural one-to-one correspondence between integers in the range from 0 and $2^n - 1$ and subsets of $\{0, \ldots, n-1\}$:

$$k \leftrightarrow \{i : i\text{-th bit of } k \text{ is } 1\}$$

Example

k	$\text{bin}(k)$	$\{i\colon i\text{-th bit of } k \text{ is } 1\}$
0	000	$\emptyset$
1	001	$\{0\}$
2	010	$\{1\}$
3	011	$\{0,1\}$
4	100	$\{2\}$
5	101	$\{0,2\}$
6	110	$\{1,2\}$
7	111	$\{0,1,2\}$

- If k corresponds to S, how to find out the integer corresponding to $S - \{j\}$ (for $j \in S$)?

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- In `C/C++`, `Java`, `Python`:
  `k ^ (1 << j)`

Outline

1. 3-Satisfiability
   - Backtracking
   - Local Search
2. Traveling Salesman Problem
   - Dynamic Programming
   - Branch-and-bound

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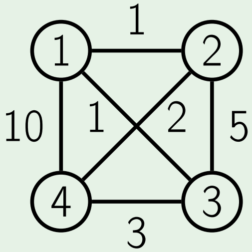
- We grow a tree of partial solutions
- At each node of the recursion tree we check whether the current partial solution can be extended to a solution which is better than the best solution found so far
- If not, we don't continue this branch

Example: brute force search



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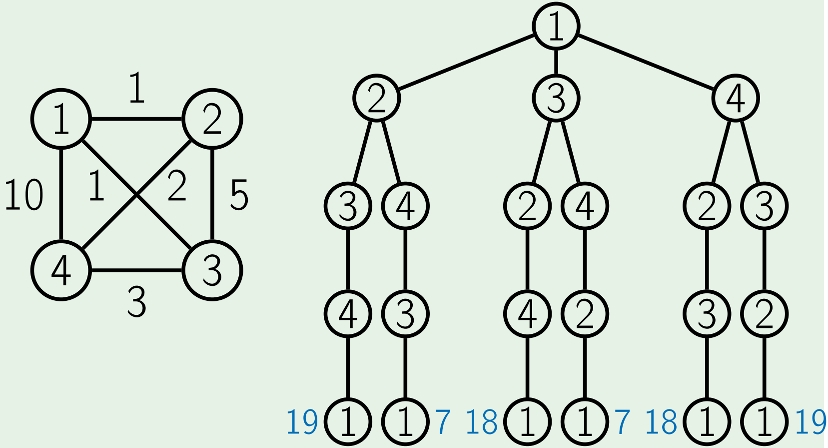
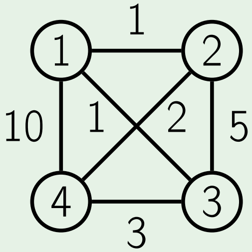


Example: pruned search



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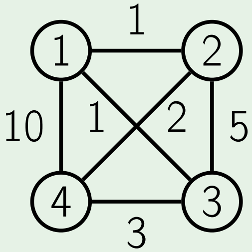

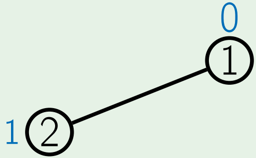

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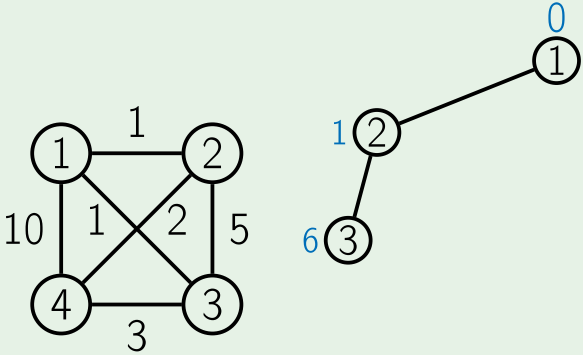

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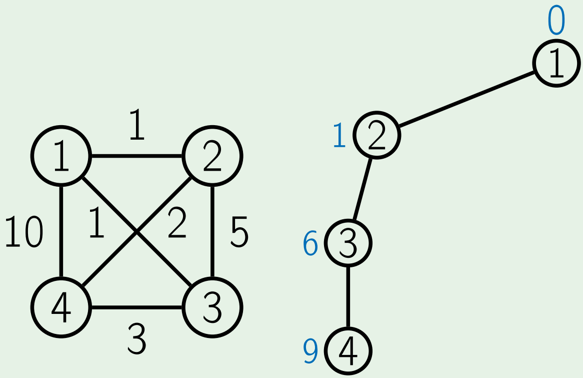

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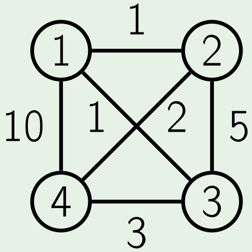
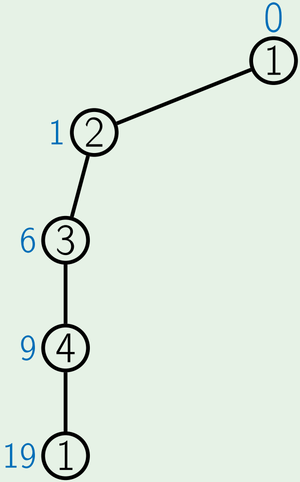
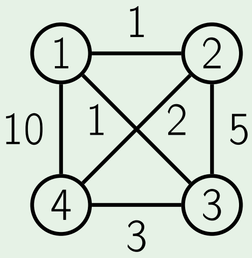
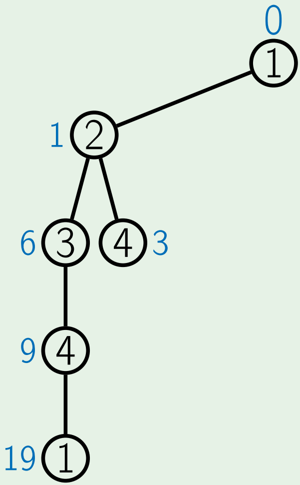




best so far: 19

Example: pruned search





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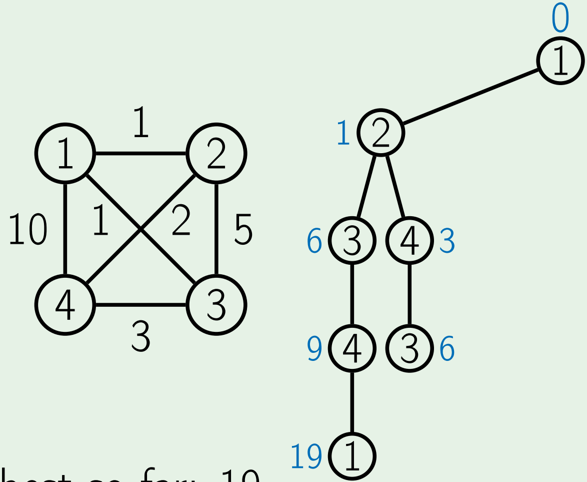

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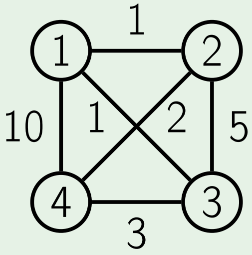

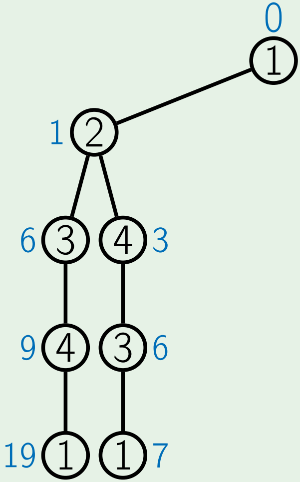

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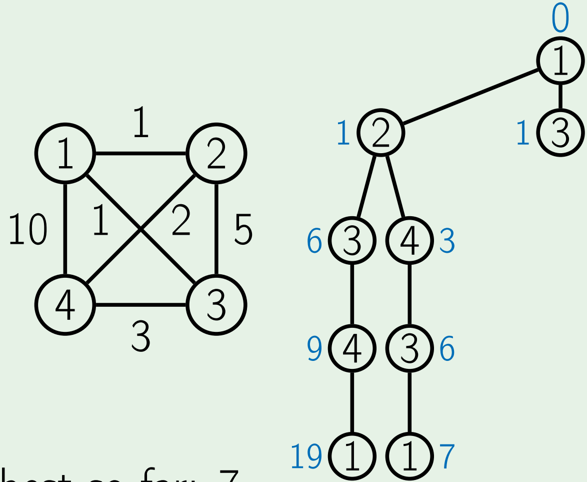
Example: pruned search



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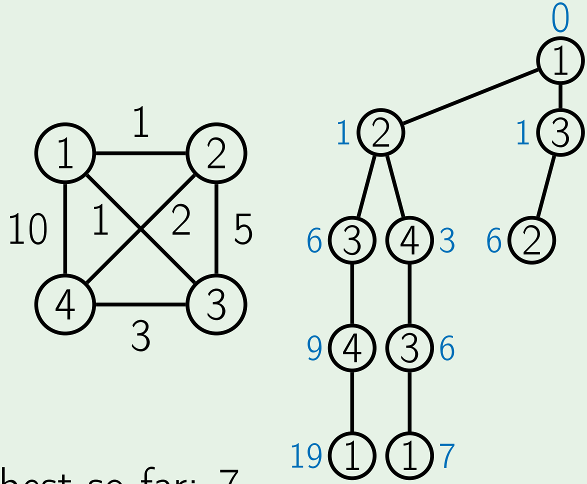

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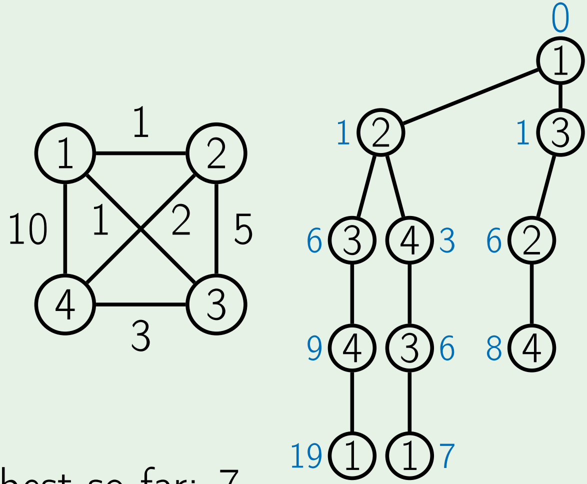

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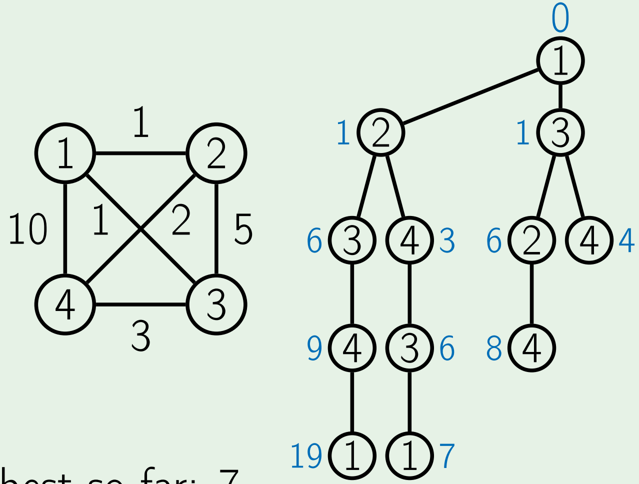

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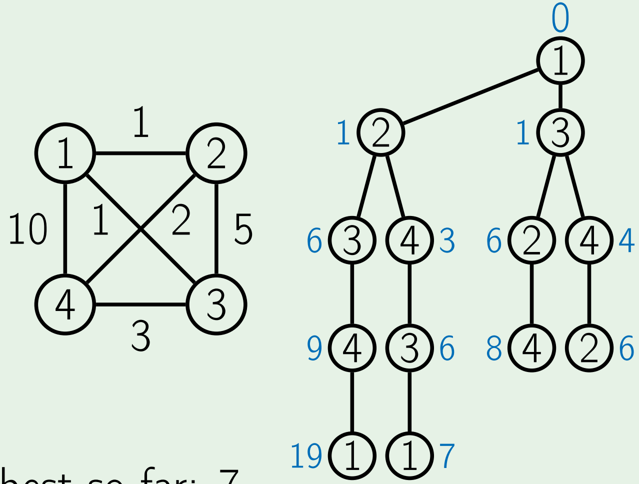

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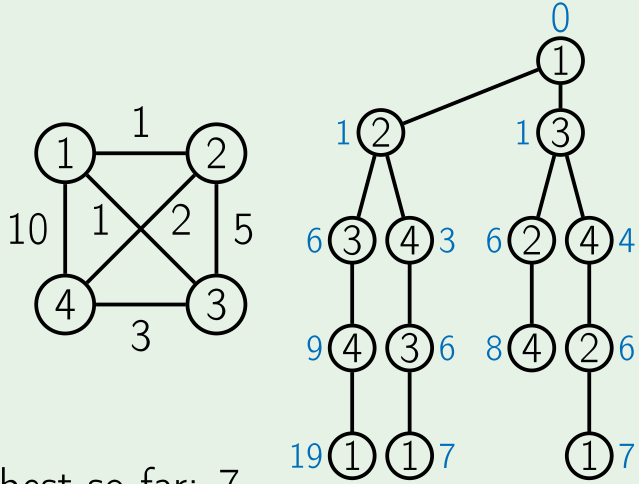

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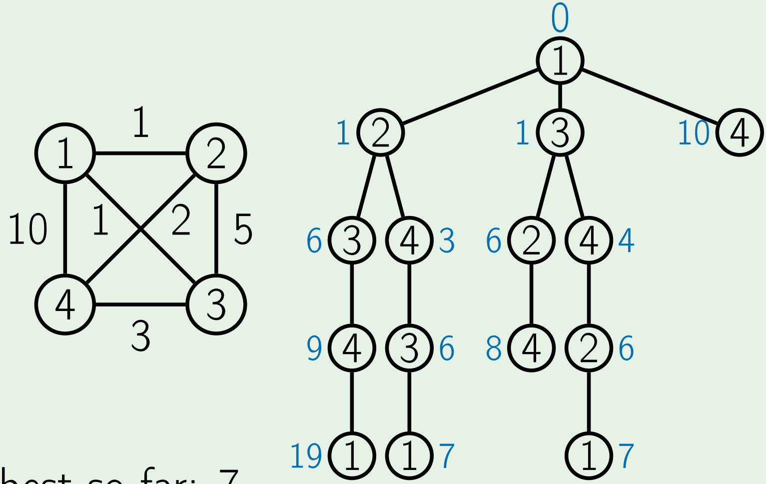

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- We used the simplest possible lower bound: any extension of a path has length at least the length of the path

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- Modern TSP-solvers use smarter lower bounds to solve instances with thousands of vertices

Example: lower bounds (still simple)

The length of an optimal TSP cycle is at least

- $\frac{1}{2}\sum_{v \in V}$(two min length edges adjacent to v)

Example: lower bounds (still simple)

The length of an optimal TSP cycle is at least

- $\frac{1}{2}\sum_{v \in V}$(two min length edges adjacent to v)
- the length of a minimum spanning tree

Next time

Approximation algorithms: polynomial algorithms that find a solution that is not much worse than an optimal solution